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**Small resolutions and non-liftable Calabi-Yau threefolds. (English summary)**

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In their papers, M. Hirokado [Tohoku Math. J. (2) **51** (1999), no. 4, 479–487; [MR1725623 \(2000m:14044\)](#)] and S. Schröer [Compos. Math. **140** (2004), no. 6, 1579–1592; [MR2098403 \(2005i:14051\)](#)] constructed examples of projective Calabi-Yau threefolds in characteristic 2 and 3 that have no liftings to characteristic zero. The question arises whether there exist non-liftable examples in higher characteristic.

In the present paper, the authors construct further examples of non-liftable Calabi-Yau threefolds, most notably a rigid Calabi-Yau threefold in characteristic 3 and a projective Calabi-Yau threefold in characteristic 5 with an obstructed deformation.

The authors' method exploits a remarkable feature of birational geometry in dimension  $\geq 3$ , namely the appearance of singularities that have a small resolution, that is, admit a map  $\pi: Y \rightarrow X$  such that the exceptional set has codimension  $\geq 2$  in  $Y$ . The key example of the ordinary double point in dimension three was described by M. F. Atiyah [Proc. Roy. Soc. London. Ser. A **247** (1958), 237–244; [MR0095974 \(20 #2472\)](#)]. It admits two different small resolutions that contract a single rational curve with normal bundle  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ . Such a rational curve is a stable submanifold in the sense that it lifts to any deformation of the ambient variety. As a consequence, blowing down deformations of such small resolutions give deformations of the variety which remain singular. This is very different from the geometry related to the ordinary double point in dimension two.

The authors use this peculiar property of the three-dimensional ordinary double point to give the following criterion for varieties in characteristic  $p$  to have no lifting to characteristic zero.

**Criterion.** Let  $\mathcal{X}$  be a flat and proper scheme over  $S = \operatorname{Spec}(A)$  of relative dimension 3,  $A$  a complete Noetherian domain with residue field  $k$  and fraction field  $K = Q(A)$ . Assume that:

- (1) The generic fibre  $X_\eta := \mathcal{X} \otimes_A K$  is smooth.
- (2) The special fibre  $X := \mathcal{X} \otimes_A k$  is rigid with nodes as singularities.

Let  $\pi: Y \rightarrow X$  be a small resolution. Then  $Y$  does not lift to  $S$ .

As a corollary, under the above assumptions, if  $A = \mathbb{Z}/p\mathbb{Z}$ , then  $Y$  has no lift to characteristic zero.

The above criterion can be applied to certain rigid Calabi-Yau threefolds that can be obtained as the resolution of a double cover of  $\mathbb{P}^3$ , ramified over a reducible hypersurface of degree eight and fibre products of rational elliptic surfaces.

Reviewed by [Lei Yang](#)

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